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LINEAR COMPLEMENTARY INEQUALITIES FOR ORDERS IN ANALYTIC GEOMETRY (ŁOJASIEWICZ INEQUALITIES AND STRONG APPROXIMATION THEOREMS)

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We propose a unified view of several topics on singularity, local rings and function theory and point out some relations among them. They are all expressed by *linear complementary inequalities* between some kind of orders.

Introduction

The order $\nu(f)=\nu_{\xi}(f)$ of an analytic function germ f at $\xi\in\mathbf{C}^n$ is defined as the degree of the leading homogeneous term of the Taylor expansion of f at ξ. We can generalize this to analytic function germs f at a singularity (X,ξ). Some standard operations yield trivial inequalities for orders. For example, the order of a product fg is not less than the sum of the orders of f and g. If (X,ξ) is integral (=reduced and irreducible), the inequality has *linear complementary inequality* (LCI: If an inequality $P\leqq Q$ is given, by linear complementary inequality we mean an inequality of the form $Q\leqq aP+b$). The infimums of coefficients of the LCIs are invariants that measure the badness of the operation applied.

Through blowings up, such a result about singularities are related to the geometry around the exceptional sets or Moishezon subspaces and further the analysis of polynomial functions on affine varieties.

It is an elementary fact that the absolute value of an analytic function is locally estimated from above by a multiple of a power of the distance from its zero-locus. Its complementary inequality always holds and is called *Łojasiewicz inequality*. If we take logarithm, it is an LCI.

There is its ultrametric analogue by Greenberg, namely, the coefficient field $\mathbf{R}$ can be replaced by an excellent Henselian discrete valuation ring. Artin has generalized Greenberg's theorem to a certain kind of rings. Such a property of a ring is called the *strong approximation property* (SAP). Artin's result is a complementary inequality without linearity. Recently linearization of SAP is attempted by a few mathematician.

In the last part we introduce a little different kind of inequalities. It is concerned with the exterior derivation in the de Rham complex of a contractible analytic algebra.

It has been a convince of the author that, if we are given an order function, the first thing is to take up a most ovbious inequality and seek for its LCI.

1. Order

Let k denote the field $\mathbb{C}$ or $\mathbb{R}$. The *order* $\nu_\xi(f)$ of

$$f=\sum c_\alpha(x-\xi)^\alpha\in k\{x-\xi\} \quad (x=(x_1,\ldots,x_n))$$

at $\xi=(\xi_1,\ldots,\xi_n)$ is defined by

$$\nu_\xi(f^\sim)\equiv\min\{|\alpha|: c_\alpha\neq 0\}$$

$$(\alpha\equiv(\alpha_1,\ldots,\alpha_n),\quad \alpha\equiv|\alpha_1|+\ldots+|\alpha_n|).$$

If $X_\xi=(X,\xi)$ is the germ at ξ of a k-analytic subspace (or a k-analytic subset) of k^n, $\mathcal{O}_{X,\xi}$ denotes the k-algebra of germs at ξ of analytic functions on X (restrictions of analytic functions in a neighborhood of ξ in k^n). Let $I\subset k\{x-\xi\}$ denote the ideal of all $f^\sim\in k\{x-\xi\}$ whose restrictions to X vanish in neighborhoods of ξ. Then we have $\mathcal{O}_{X,\xi}\cong k\{x-\xi\}/I$. An algebra isomorphic to $\mathcal{O}_{X,\xi}$ is called an *analytic local algebra*. We put $A\equiv\mathcal{O}_{X,\xi}$, $\mathfrak{m}\equiv(x_1-\xi_1,\ldots,x_n-\xi_n)A$ (the unique maximal ideal of A). We define the *order* of $f\in A$ by

$$\nu_\xi(f)\equiv\nu(f)\equiv\sup\{p: f\in\mathfrak{m}^p\}$$

$$\equiv\sup\{\nu_\xi(f^\sim): f^\sim\in k\{x-\xi\}\text{ is a representative of } f\}$$

and *reduced order* by

$$\overline{\nu}(f)\equiv\lim_{p\to\infty}\nu(f^p)/p.$$

Example 1.1. Let us put $X=\{y^2-x^3=0\}\subset k^2$, $f\equiv(y^3)_0$ (the suffix $_0$ indicate the germ at 0). Since $y^3=x^3y+(y^2-x^3)y$, $\nu(f_0)=4$. Since $\nu(f_0^{2n})=9n$ and $\nu(f_0^{2n+1})=9n+4$, we have $\overline{\nu}(f_0)=(9/2)$.

These definitions of ν and $\overline{\nu}$ are applicable to any local ring $(A,\mathfrak{m})$. Our first LCI is the following.

$\boxed{\text{Theorem 1.2.}}$ (Rees [R1]) A local ring A is analytically unramified (i.e. the completion is reduced), if and only if

(CI_0) $\quad \exists\, b \geqq 0 \quad (\nu(f) \leqq)\ \overline{\nu}(f) \leqq \nu(f) + b \quad (f \in A)$.

(Hereafter parenthesized inequality means a trivial one, to which the word "complementary" refers.)

2. LCI for orders at a point

Following Gabrielov [G], let us define the generic rank of a germ of analytic map $\Phi : Y \longrightarrow X$ at η by

$$\mathrm{grk}_\eta \Phi \equiv \inf\{\text{topological dimension of the images of neighborhoods of } \eta \in Y\}/\varepsilon$$

($\varepsilon = 1$ in the real case, $\varepsilon = 2$ in the complex case).

We could have defined the generic rank using only algebraic terms (those of universal finite differential modules).

$\boxed{\text{Theorem 2.1.}}$ (cf. [I2],[I3]) Suppose that X_ξ is a germ of a analytic space over $\mathbf{C}$. Then the condition that X_ξ is integral is equivalent to any one of the following.

(CI_1) $\quad \exists\, a, b \in \mathbf{R},\ \forall f, g \in A\ (\equiv O_{X,\xi})$:

$$(\nu(f) + \nu(g) \leqq)\ \nu(fg) \leqq a\{\nu(f) + \nu(g)\} + b.$$

(CI_2) For any germ $\Phi_\eta : Y_\eta \longrightarrow X_\xi$ of analytic map with $\mathrm{grk}_\eta \Phi = \dim X_\xi$, we have

$$\exists\, a, b \in \mathbf{R},\ \forall f \in A:\ (\nu(f) \leqq)\ \nu(f \circ \Phi) \leqq a\nu(f) + b.$$

(CI_3) For any subanalytic set ([Hi]) $S \subset X$ with $\dim S_\xi = \varepsilon \cdot \dim X_\xi$ we have

$$\exists\, a, b \in \mathbf{R},\ \forall f \in A:\ (\nu(f) \leqq)\ \mu_S(f) \leqq a\nu(f) + b,$$

where $\mu_S(f)$ denote the order of $f|S$ with respect to the Euclidean distance.

Here the regular cases of (CI_1) is trivial $(\nu(fg) = \nu(f) + \nu(g))$ and that of (CI_3) follows from a more precise result [Sp] of Spallek. The inequality in (CI_2) implies $\mathrm{grk}_\eta \Phi = \dim X_0$ conversely ([I4]). Moreover, this inequality is equivalent to the condition that the homomorphism φ between local rings induced by Φ has a closed image with respect to the Krull topology ([BZ] cf.[I5]). Tougeron [To1] obtained an interesting proof of (CI_2), using a nice supplement to Gabrielov's theorem on convergence of formal functions. As for

(CI_1), Rees has obtained the following generalization.

Theorem 2.2. ([R2]) Let A be a local ring. Then the completion A is integral if and only if (CI_1) holds.

Theorem 2.3. ([I6]) Suppose that X_ξ is an integral germ of a complex space. If $\mathcal{C}$ is "a non-negligible family" of curves on X through ξ. Then

$$\exists\, a, b \in \mathbf{R},\ \forall f \in O_{X,\xi}:\ (\nu(f) \leqq)\ \inf_{\Gamma \in C} \nu(f \circ \Gamma) \leqq a\nu(f) + b.$$

3. Vanishing order along a subspace

Let $(X, \mathcal{O}_X)$ be a complex space, $\mathcal{I}$ a coherent ideal sheaf and f a section of $\mathcal{O}_X$. We put

$$\nu_{\mathcal{I},\xi}(f) \equiv \sup\{p:\ f_\xi \in \mathcal{I}_\xi^{\,p}\},\quad \overline{\nu}_{\mathcal{I},\xi}(f) \equiv \lim_{p\to\infty} \nu_{\mathcal{I},\xi}(f^p)/p.$$

Theorem 3.1. ([I7]) Let $S \subset X$ be a complex subspace defined by a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Suppose that S is a Moishezon space and X is integral along S. Then, if $f \in \Gamma(S, O_X)$ vanishes at a point of S with high order with respect the maximal ideal, so is f along entire S, i.e.

$$\forall \xi \in S,\ \exists\, a,\ b \in \mathbf{R}:$$
$$[f \in \Gamma(S, O_X),\ \nu(f_\xi) \geqq ap + b,\ \eta \in S] \Longrightarrow \overline{\nu}_{I,\eta}(f) \geqq p.$$

The following is the generalization of the trivial fact that, if 0 is a d-ple root of $f \in \mathbf{C}[x]$, then $\deg f \geqq d$.

Theorem 3.2. ([I7]) Let S be an integral Moishezon space, D a Cartier divisor and L(D) the space of meromorphic functions on S whose pole divisor is at most D. Then for any $\xi \in S$ and for any irreducible component $Y_\xi \subset X_\xi$,

$$\exists\, a \in \mathbf{R}:\ f \in L(dD) \setminus \{0\} \Longrightarrow \overline{\nu}(f_\xi \mid Y_\xi) \leqq ad.$$

(3.1) and (3.2) are mutually equivalent and they are also equivalent to (CI_1) of (2.1). (3.2) yields the following characterization of algebraic set germ.

Theorem 3.3. ([I8]) Let S_ξ be an irreducible germ of analytic subset at $\xi \in \mathbf{R}^n$ (or $\mathbf{C}^n$). Then the following conditions are equivalent.

(i) S_ξ is an analytic irreducible component of an algebraic set.

(ii) $\exists\, a \in \mathbf{R}$: $f \in \mathbf{C}[x]$, $\deg f = d \Longrightarrow \overline{\nu}(f|S_\xi) \leqq ad$.

By this we may say that

$$\sup\{\log \nu(f|S_\xi)/\log \deg f : f \in \mathbf{C}[x]\}$$

measures the *transcendence* of the embedded singularity $S_\xi \subset \mathbf{C}^n$.

4. Łojasiewicz inequality

Let f be an analytic function on an open subset U of $\mathbf{C}^n$ or $\mathbf{R}^n$ and $K \subset U$ a compact subset. Then it is easy to see that

(*) $\exists\, a', b' \in \mathbf{R}$, $\forall x \in K$: $|f(x)| \leqq b' \cdot \mathrm{dist}(x, f^{-1}(0))^{a'}$.

Lejeune-Jalabert - Tessier [LT] characterized sup a′ using integral dependence to an ideal sheaf. Bochnak-Risler [BR], Risler [Ri] and Fekak [F1] treated the real case. These papers verified rationality of sup a′. This is related to rationality of the reduced order $\overline{\nu}$ (see [LT]), which is asked by Samuel and settled by Rees and Nagata independently.

The complementary inequality to (*) is needed in the theory of Schwartz distribution. The statement is the following.

$\boxed{\text{Theorem 4.1.}}$ ([Hö], [Łoj]) Let f be an analytic function on an open subset U of $\mathbf{C}^n$ or $\mathbf{R}^n$ and $K \subset U$ a compact subset. Then

(**) $\exists\, a, b \in \mathbf{R}$, $\forall x \in K$: $|f(x)| \geqq b \cdot \mathrm{dist}(x, f^{-1}(0))^{a}$.

The logarithm of (**) is an LCI of that of (*). It is confusing that the term *Łojasiewicz exponent* of f implies both sup a′ in (*) and inf a in (**). (If we consider f and the distance function on the same level, they are merely mutual inverse (cf. [F2]).)

The polynomial case of (4.1) was proved by Hörmander and the general case by Łojasiewicz. It implies that if $|f(x')|$ is small, there exists a solution x of $f(x)=0$ near x' or that

an approximate solution x' *is near to an actual one* x.

There exist many versions of (**). In his course of study of determinacy of function germs, Kuo [K] obtained an effective result on inf a for plane curves, using Newton diagram. Schappacher [Sc] obtained LCI for rigid analytic equations with unknowns sought in the valuation ring of a complete field with non-archmedean valuation. Bollaerts [B] obtained an effective local result for almost all algebraic equations with unknowns sought in $\mathbf{R}$ and $\mathbf{Q}_p$ also using Newton diagram.

In the polynomial case, global Łojasiewicz inequality attracts many experts in transcendence theory, complex analysis and algebraic geometry. For example, polynomials with coefficients in $\mathbf{Z}$ are treated by Brownawell [Br] and those with coefficients in $\mathbf{C}$ are treated by Ji- Kollár-Shiffman [JKS]. They are seeking for sharp effective bounds of exponents etc. Similar topics are effective Nullstellensatz and effective bound for division problem (cf. [BY] for literature). Tessier has written a thoughtful survey [T] on these topics.

Bierstone-Milman exhibit a very simple proof of Łojasiewicz inequality for subanalytic functions. Fekak [F2] treated semialgebraic case and obtained rationality of the exponent for global Łojasiewicz inequality and a nice property of parametrized families. Tougeron [To2],[To3] treated wider classes of functions. The classes include the exponential extension of polynomial rings, which is studied by van den Dries and Wilkie. Loi [Lo] announced similar results.

A weaker condition than Łojasiewicz inequality is introduced by Lengyel [Le] and used to state the condition for differentiability of power roots of non-negative smooth functions.

5. Inequality for functional equations

We need not restrict ourselves to the equations with unknowns sought in number fields.

Theorem 5.1. (Greenberg [Gr]) Let R be an excellent Henselian discrete valuation ring (DVR) with the maximal ideal $\mathbf{p}$. If $f\equiv(f_1,\dots f_m)\in R[x_1,\dots,x_n]^m$, then $\exists\, a,b\in\mathbf{R}$:

$$X'_1,\dots,X'_n\in R,\ \ f(X'_1,\dots,X'_n)\equiv 0 \mod \mathbf{p}^{ak-b}$$
$$\Longrightarrow\ \ \exists\, X_1,\dots,X_n\in R,\ \ f(X_1,\dots,X_n)=0,\ \ X'_i\equiv X_i \mod \mathbf{p}^k.$$

This implies also that an approximate solution is near to an actual one. (5.1) is the origin of the famous strong approximation theorem (SAP) of Artin [A] on polynomial equations with unknowns sought in the Henselization at a point of a polynomial ring over a field. The following is a generalization of Artin's SAP. (Artin's SAP is the case when f is a system of polynomials.)

$\boxed{\text{Theory 5.2.}}$ (Wavrik [W1]) Suppose that k is a field of characteristic 0 complete with respect to a valuation and $f \equiv (f_1, \ldots f_m) \in k\{x, y\}[z]^m$, ($x \equiv (x_1, \ldots, x_n)$, $y \equiv (y_1, \ldots, y_p)$, $z \equiv (z_1, \ldots, z_r)$). Then, for any $t \in \mathbf{N}$, there exists $\beta(t) \in \mathbf{N}$ such that, if $Y' \in k[\![X]\!]^p$ with $Y'(0)=0$ and $Z' \in k[\![X]\!]^r$ satisfy $f(x, Y', Z') \equiv 0 \bmod x^{\beta(t)}$, there exist $Y \in k\{x\}^p$ with $Y(0)=0$ and $Z \in k\{x\}^r$ which satisfy $f(x, Y, Z)=0$ and $Y' \equiv Y$, $Z' \equiv Z \bmod x^{\alpha}$.

The least function $\beta(t)$ that satisfies the condition as above is called the *Artin function* for analytic equation $f(x, Y, Z)=0$. By the work of Pfister-Popescu [PP] it is known that SAP holds for equations with unknowns sought in complete local rings also (cf. [DL], [N]). These treat very general equations but lack "linearity" except (5.1). Lascar [L] has shown that $\beta(t)$ in original Artin's SAP is recursive.

An important kind of analytic equation arises as the condition constraining curves to an analytic singularity. Then the unknowns are sought in $\mathbf{C}\{t\}$ or $\mathbf{C}[\![t]\!]$. (This is related to Nash's theory on singularities. He began to study a singularity through the set *H* of formal curves constrained to it. *H* is considered as the inverse limit of the algebraic varieties which consists of truncated curves ([GL]).) Wavrik [W2], Lejeane-Jalabert [L], Ellias [E], Hickel [H] and Gonzalez-Sprinberg - Lejeane-Jalabert [GL] obtained $\beta(t)$ for such analytic equations. Their results are effective and often best.

As for LCI for the equation with unknowns sought in a higher dimensional local ring, we know little. The most simple nontrivial example is (CI_1) of (2.2). To see this suppose that

(A, $\mathbf{m}$) is a local ring with a integral completion and consider the equation $XY=0$ over A. Then (2.2) implies that

$$(s', t') \in A\times A,\ s't' \equiv 0 \bmod \mathbf{m}^{2ak+b}$$
$$\Longrightarrow \exists\,(s,t)\in A\times A:\ s'\equiv s,\ t'\equiv t \bmod \mathbf{m}^{k},\ st=0.$$

Here (s', t') is an approximate solution and (s,t) an actual solution and they are near. We show another example.

Example 5.3. ([I3], (5.1)) Take a prime number p and suppose that $u\in \mathbf{C}\{x\}$ $(x=(x_1,\ldots,x_n))$ is not a p-th power in $\mathbf{C}\{x\}$. Then the equation $S^p-uT^p=0$ over $\mathbf{C}\{x\}$ admits an LCI. Indeed, this equation has a unique solution (0,0) and

$$\exists\, a,b\in \mathbf{R}:\ f^p-ug^p\equiv 0 \bmod m^{ak+b} \Longrightarrow f\equiv 0,\ g\equiv 0 \bmod m^k.$$

As an answer to Popescu's problem, Spivakovsky [Spv] has shown an example of a Henselian pair for which an analogue of the strong approximation theorem fails, i.e. even a nonlinear $\beta(t)$ does not exists.

6. LCI for exterior derivation

Let A be a ring, $\mathbf{I}\subset A$ an ideal and

$$\Theta^{\cdot}\equiv \{\Theta^{-1}\xrightarrow{d}\Theta^{0}\xrightarrow{d}\Theta^{1}\xrightarrow{d}\Theta^{2}\xrightarrow{d}\cdots\}$$

a complex of A-modules. We can define the order of $\omega\in\Theta^p$ $(p\geqq 0)$ using the filtration $\{\mathbf{I}^k\Theta^p\}_{k\geqq 0}$:

$$\nu_{\mathrm{I}}(\omega)=\sup\{k:\ \omega\in \mathbf{I}^k\Theta^{\cdot}\}.$$

Consider the following conditions for a, $b\in\mathbf{R}$ (cf. [F]):

$$(\mathrm{O1})^p \qquad \omega\in\Theta^p\cap d^{-1}(0)$$
$$\Longrightarrow \exists\,\theta\in\Theta^{p-1},\ \omega=d\theta,\ \nu_{\mathrm{I}}(\omega)\leqq a\nu_{\mathrm{I}}(\theta)+b;$$

$$(\mathrm{O2})^p \quad \omega\in\Theta^p \Longrightarrow \exists\,\xi\in\Theta^{p-1},\ \nu_{\mathrm{I}}(d\omega)\leqq a\nu_{\mathrm{I}}(\omega-d\xi)+b.$$

The latter is in an LCI modulo the space of exact forms. It follows that exterior derivation is an open mapping onto the image (=the space of exact forms). The following is easy to see.

$\boxed{\text{Lemma 6.1.}}$ For p=0, 1, 2, ..., we have the following.

(i) $(O1)^p \Longrightarrow H^p(\Theta^\cdot)=0$.

(ii) $(O2)^p$ and $(\bigcap \mathbf{I}^k \Theta^p=0) \Longrightarrow H^p(\Theta^\cdot)=0$.

(iii) $(O1)^{p+1}$ and $H^p(\Theta^\cdot)=0 \Longleftrightarrow (O2)^p$ and $H^{p+1}(\Theta^\cdot)=0$.

Let

$$\Omega^\cdot(A)\equiv\{\mathbf{C}\longrightarrow A\longrightarrow \Omega^1(A)\longrightarrow \Omega^2(A)\longrightarrow \cdots\}$$

be the analytic de Rham complex (the complex of Pfaffian forms on A) ([GR],[Rei]), where $\mathbf{C}\longrightarrow A$ denotes the canonical injection. The condition $\bigcap_{k\in\mathbb{N}} \mathbf{I}^k\Omega^p=0$ is satisfied by this complex. It is obvious that $\nu_I(\omega)\leqq\nu_I(d\omega)+1$. Sasakura found the following.

$\boxed{\text{Theorem 6.2.}}$ ([Sas], cf. [Fu]) If $A\equiv\mathbf{C}\{x\}$ $(x=(x_1,\ldots,x_n))$, the conditions $(O1)^p$ and $(O2)^p$ hold for

$$\Theta^p\equiv\Omega^p(A),\quad (\exists a,\ b\in\mathbf{R};\ p=0,1,2,\ldots).$$

Their results are stronger than stated here in that they treat Pfaffian forms *on a neighborhood.*

Anyone who learned the elementary calculus understands that

$$\nu_0(f-f(0))=\inf\{\nu_0(\partial f/\partial x_1),\ldots,\nu_0(\partial f/\partial x_n)\}+1$$

for $f\in\mathbf{C}\{x\}$. This can be generalized as follows.

$\boxed{\text{Theorem 6.3.}}$ ([I1]) If A is holomorphically contractible into an analytic local algebra with embedding dimension n (in the sense of Reiffen [Rei]), then the conditions $(O1)^p$ and $(O2)^p$, with "$\leqq$" replaced by "$=$", hold for

$$\Theta^p\equiv\Omega^p(A),\quad p=n,n+1,n+2,\ldots,$$
$$a=1,\ b=-1,\ I=(\text{the maximal ideal}).$$

(6.2) and (6.3) are sharper than the Poincaré lemma. The same assertion as (6.3) holds for AC-contractible formal algebras (AC: absolutely continuous, cf. [I1]).

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